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WEIGHTED PCM

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Subject: Weighted PCM

Summary

A comparison is made between weighted PCM and conventional PCM from a fidelity viewpoint. For equal average transmitted power, it is shown that a weighted PCM system is capable of reducing the analog error associated with the reconstructed telemetry message by approximately an order of magnitude (equivalent to a signal-to-noise ratio gain of 2 db) at moderate rf signal-to-noise ratios. Optimum and more practical, near-optimum systems are analyzed.

Introduction

The elements of conventional PCM have been previously discussed wherein, by the processes of sampling and quantization, an analog waveform may be represented by a set of discrete values. These values are usually expressed as binary numbers, and their electrical equivalents (pulse sequences) are transmitted over a noisy binary channel. The pulses normally have equal amplitudes and widths, and the digit error probability after decoding at the receiver is the same for all orders in the binary sequence. In the transmission of numbers, however, the orders of the pulse sequence have varying degrees of importance with respect to the fidelity of the reconstructed analog waveform. That is, for binary numbers the value or weight of the ith pulse in a sequence is 2^{i-1}, which is seen to increase by powers of two as the order i increases. An error in a high order pulse has considerably more deleterious effect than an error in a lower order pulse. Therefore from a fidelity viewpoint, it is beneficial to modify the PCM process such that account is made of the importance of a pulse position. The weighted PCM system analyzed in this report performs this function.

Some previous work has been reported on weighted PCM with respect to the criteria of utility and signal-to-noise ratio.[2,3] In this report a criterion has been chosen which is believed more meaningful to the telemetry designer. The analyses performed give the specification of an optimum weighted PCM system from the standpoint of minimum percentage rms or absolute analog error in the reconstructed waveform. Consideration is given to errors caused by channel noise and errors due to quantization. A more practical near-optimum weighted PCM system is also analyzed and comparisons are made with conventional PCM.

Analysis

The system model chosen is a binary symmetric PCM channel, perturbed by additive gaussian noise, with coherent detection on a bit basis. The average error probability in the detection of the individual pulses in the PCM train is a function of both pulse amplitude and rms noise present and is determined from*

$$p_i = \frac{1}{\sqrt{2\pi}\sigma} \int_{a_i}^{\infty} e^{-n^2/2\sigma^2} \, dn \tag{1}$$

$$= \frac{1}{2} [1 - \Phi(a_i/\sqrt{2}\sigma)] \tag{2}$$

where

$$\Phi = \frac{2}{\sqrt{\pi}} \int_{0}^{a_i} \exp - x^2 \, dx \tag{3}$$

*See Glossary for definition of symbols used, p. 21.

In ordinary, unweighted PCM each pulse is transmitted at the same amplitude and therefore all the p_i's are equal, $i = 1, 2, ---, n$, in a code word. This is the case even though each pulse position has a different weighting with respect to the analog voltage that the PCM word represents. In a weighted PCM system, optimally adjusted, all the pulses are transmitted at different amplitudes such that the p_i's decrease with increasing order of the digit in the PCM word.

In either system the probability of a single error in the ith position of a n-digit word is

$$p_i(1,n) = p_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (1-p_j) \tag{4}$$

$$\approx p_i \tag{5}$$

for small digit error probabilities.* For unweighted PCM, since $p = p_i$, all i

$$p_i(1,n) = p \tag{6}$$

The effect of these digit errors on the original analog signal can be expressed in terms of the rms error ϵ_u or ϵ_w.

For the weighted PCM case, the mean squared analog error of a pulse group or PCM code word is

$$\epsilon_w^{\,2} = \sum_{i=1}^{n} p_i(1,n) e_i^{\,2} \tag{7}$$

*Only the effect of single errors are considered here. Double-error results are discussed later in the section.

In Appendix 1, it is shown that

$$e_i = \pm q\, 2^{i-1}$$

Thus,

$$\epsilon_w^{\,2} = q^2 \sum_{i=1}^{n} p_i\, 4^{i-1} \tag{8}$$

For purposes of comparison it is convenient to normalize ϵ_w to Q_m, the quantized signal range. Since

$$Q_m = qA_m(n) = q(2^n - 1), \tag{9}$$

$$\epsilon_{wn} = \frac{\left[\sum_{i=1}^{n} p_i\, 4^{i-1}\right]^{1/2}}{2^n - 1} \tag{10}$$

For the unweighted PCM case the mean squared analog error is

$$\epsilon_u^{\,2} = p \sum_{i=1}^{n} e_i^{\,2} \tag{11}$$

$$= q^2 p \sum_{i=1}^{n} 4^{i-1} \tag{12}$$

$$= q^2 p\, \frac{4^n - 1}{3} \tag{13}$$

Thus, the rms normalized analog error of a pulse group is

$$\epsilon_{un} = \frac{\left[p\,\frac{4^n-1}{3}\right]^{1/2}}{2^n-1} \tag{14}$$

The errors given by (10) or (14) represent that fraction of the total error which is due to channel noise. The total error is comprised of (10) or (14) plus the error due to quantization of the original analog signal. If e_q is the error voltage between the actual instantaneous signal and its quantized equivalent, the mean squared analog error due to quantization is

$${}_{q}^{2} = \frac{1}{q}\int_{-q/2}^{q/2} e_q^{2}\,de_q \tag{15}$$

$$= q^2/12 = 0.0833q^2 \tag{16}$$

and ϵ_q, the rms error is

$$\epsilon_q = q/2\sqrt{3} \tag{17}$$

Thus the total rms analog error of a pulse group normalized to Q_m is

$$\epsilon_{wtn} = \frac{\left[\epsilon_w^{2} + \epsilon_q^{2}\right]^{1/2}}{q(2^n-1)} \tag{18}$$

and

$$\epsilon_{utn} = \frac{\left[\epsilon_u^{2} + \epsilon_q^{2}\right]^{1/2}}{q(2^n-1)} \tag{19}$$

for the weighted PCM and unweighted PCM cases, respectively, where ϵ_w^2 and ϵ_u^2 are given by (8) and (13).

Another relation which is needed for the optimization procedure is the average power in a PCM code word. For an n-bit word, the average base band or video signal power is

$$P = \frac{1}{n} \sum_{i=1}^{n} a_i^2 \tag{20}$$

for weighted PCM. For unweighted PCM, all the a_i's are equal, so

$$P = a^2 \tag{21}$$

where $\quad a = a_i \text{ all } i$

The optimization problem is then to determine the pulse amplitudes a_i in the weighted PCM system such that the total rms analog error is a minimum, subject to a constraint on the average power P. Since the quantization error is a constant in (18), it is sufficient to find the a_i's which minimize (8) subject to the constraint (20). This problem is solved in Appendix 2 using Lagrange's method of undetermined multipliers. The expression for the optimum a_i's is shown to be

$$\frac{a_i^2}{\sigma^2} = \frac{P}{\sigma^2} + \frac{P/\sigma^2 - 1}{P/\sigma^2} \left[i - \left(\frac{n+1}{2}\right)\right] \ln 16 \tag{22}$$

As previously mentioned, the above analysis neglected the effects of more than one digit error per PCM word. In Appendix 3 the effects of double errors are discussed. Whereas the analysis for the optimum a_i's is difficult to perform if multiple errors are included, it is relatively simple to ascertain how much their presence modifies the previous results. A modified relation for the rms analog error (upper bound) including single and double errors was found, wherein the pulse amplitudes a_i are set equal to their optimum values computed in the single error analysis. This expression, for the weighted PCM case, is

$$\epsilon_w'^{\,2} = q^2\left\{\sum_{i=1}^{n} p_i 4^{i-1} + \sum_{j=1}^{n-1} p_j \sum_{i=j+1}^{n} p_i \left[4^{i-1}+4^{j-1}+2^{i+j-1}\right]\right\} \tag{23}$$

For the unweighted case

$$\epsilon_u'^{\,2} = q^2\left\{p\,\frac{4^n-1}{3} + p^2\left[(n+2/3)\frac{4^n}{3} + \frac{n(4^{n-1}-1)}{3} - 2^{n+1} + \frac{16}{9} + \sum_{j=1}^{n-1} j4^{j-1}\right]\right. \tag{24}$$

The corresponding normalized errors are readily obtained from

$$\epsilon'_{wn} = \frac{\epsilon'_w}{q(2^n-1)} \tag{25}$$

and

$$\epsilon'_{un} = \frac{\epsilon'_u}{q(2^n-1)} \tag{26}$$

An additional analysis was performed based on a mean <u>absolute</u> error criterion rather than a rms criterion. This is presented in Appendix 4. The procedures are very similar and it is shown that the optimum a_i's are defined by

$$\frac{a_i^2}{\sigma^2} = P/\sigma^2 + \frac{P/\sigma^2-1}{P/\sigma^2} \quad [i - \frac{n+1}{2}]\ln 4 \tag{27}$$

The normalized analog errors are given by

$$|\epsilon_{wn}| = \frac{\sum_{i=1}^{n} p_i \, 2^{i-1}}{2^n-1} \tag{28}$$

and

$$|\epsilon_{un}| = \frac{p\sum_{i=1}^{n} 2^{i-1}}{2^n-1} = p \tag{29}$$

for the weighted and unweighted PCM systems, respectively.

Results

The analytic results are plotted in several curves of percentage analog error versus rf or input signal-to-noise ratio, $P/2\sigma^2$ in db. Moreover, the results of a more practical near-optimum scheme are shown. Specifications are given for optimum and near-optimum weighting of pulse orders in a PCM code word.

In Fig. 1, a comparison of weighted and unweighted PCM systems is made, with respect to the normalized rms analog error due to channel noise alone, for code word lengths of 5, 10, and 15 and several signal-to-noise ratios. In Fig. 2 a similar comparison is made with respect to the total rms analog error, including quantization noise. In Fig. 3 the effects of including double errors are shown for a code-word length of 10. The significant results are:

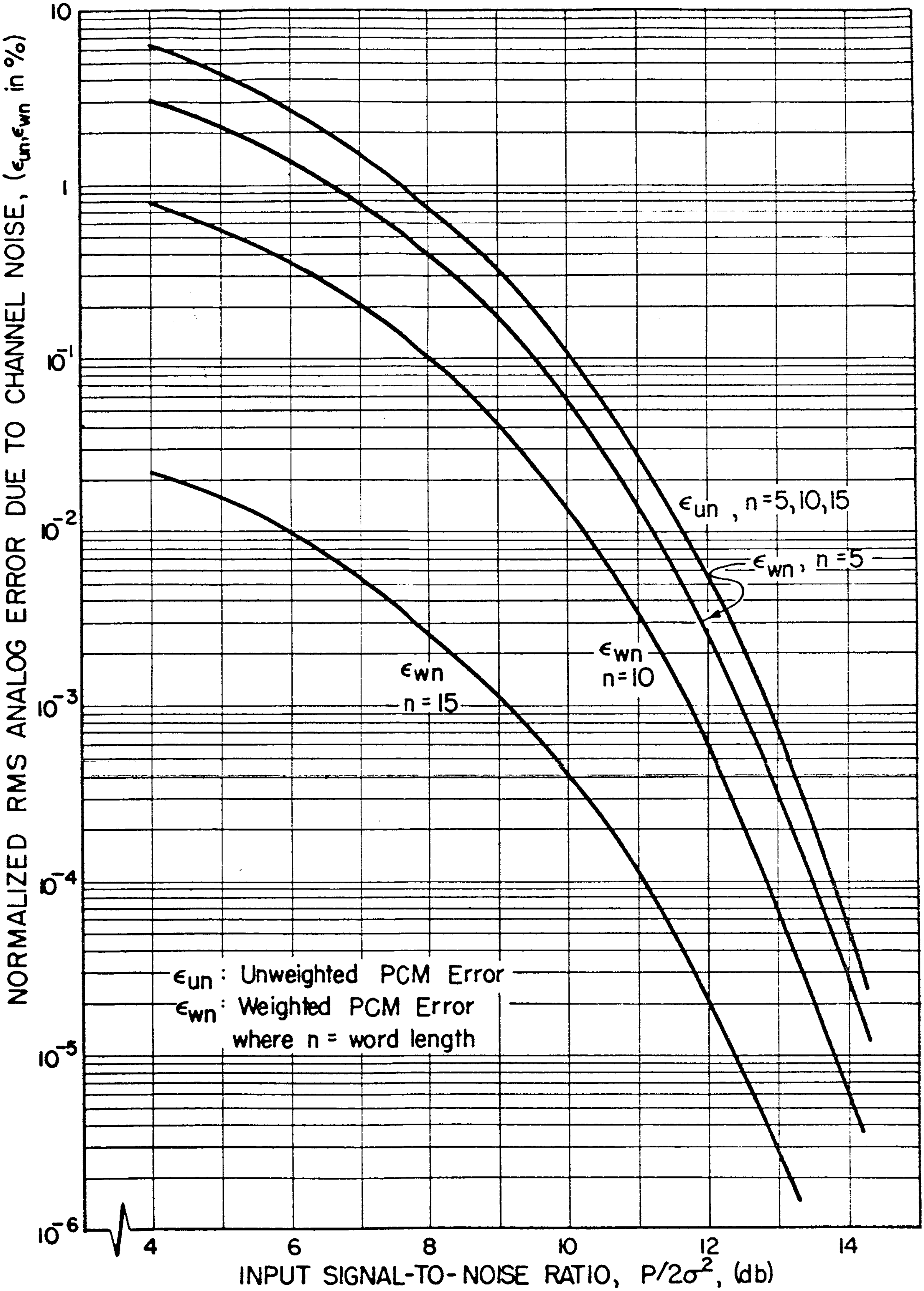

Fig. 1 Comparison of normalized rms analog errors, due to channel noise, of weighted and unweighted PCM systems for several code lengths and signal-to-noise ratios.

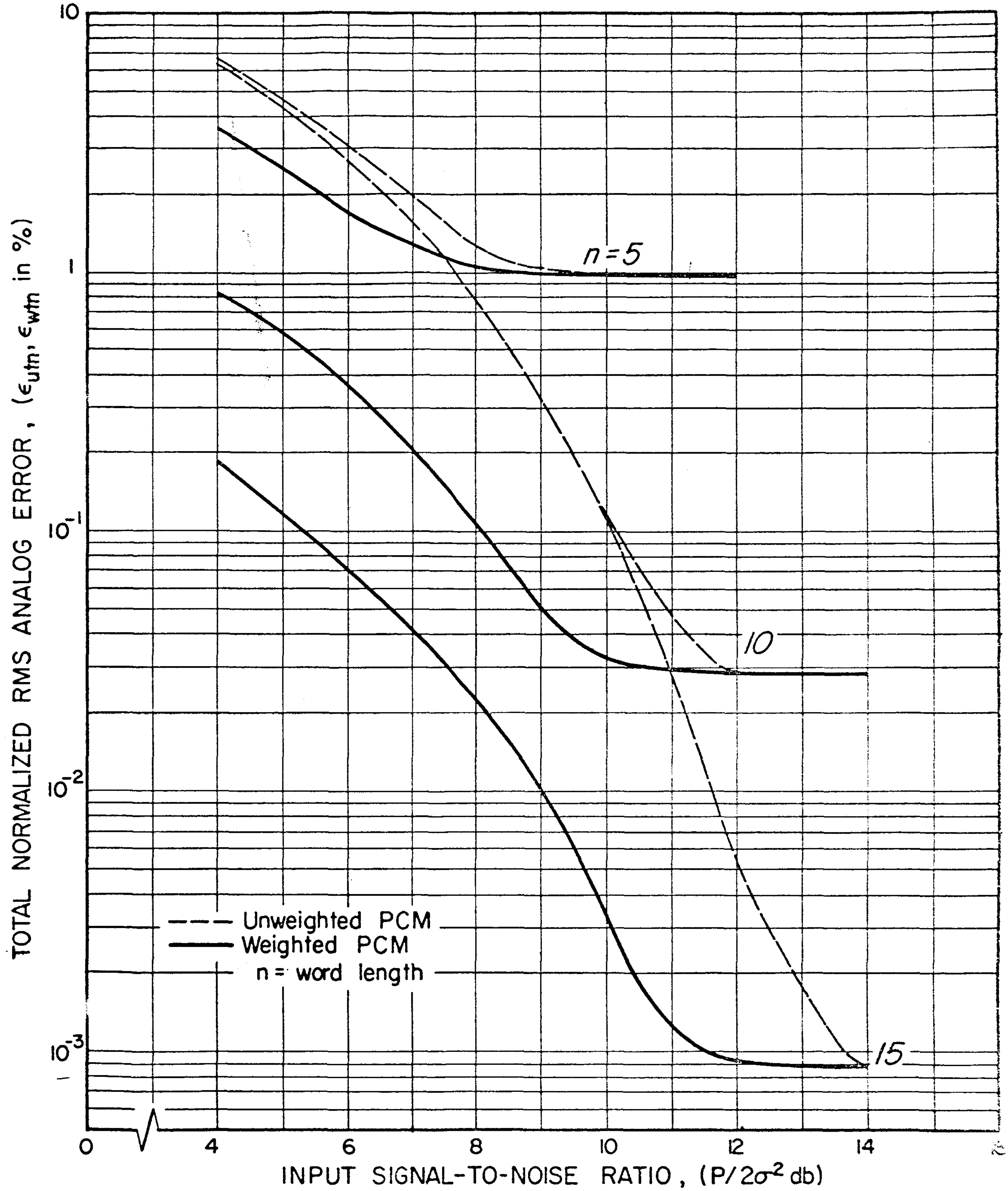

Fig. 2 Comparison of total normalized rms analog errors of weighted and unweighted PCM systems for several code lengths and signal-to-noise ratios.

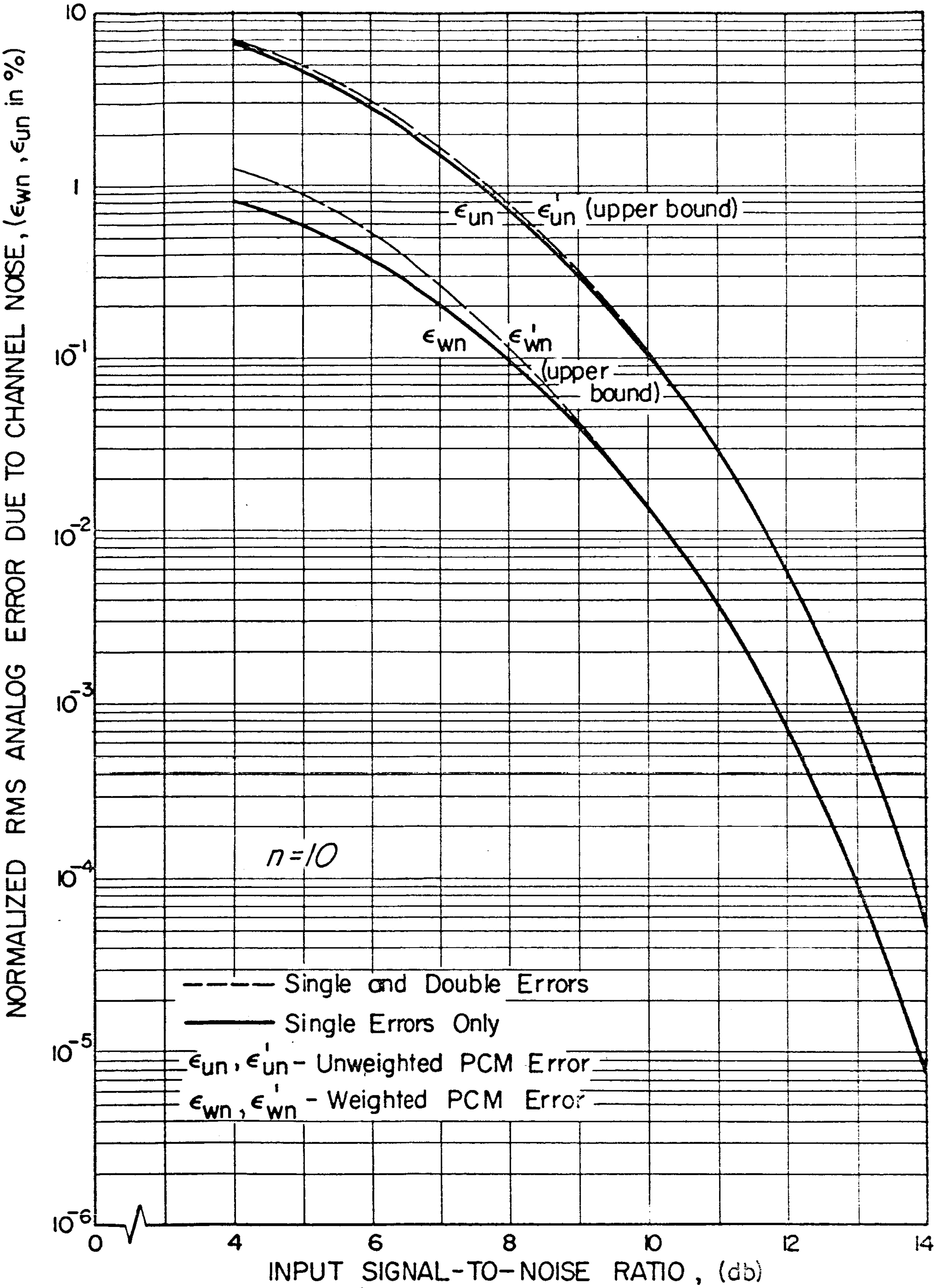

Fig. 3 Comparison of normalized rms analog errors, due to channel noise, of weighted and unweighted PCM systems, including the effect of double errors, for a code length n=10, and several signal-to-noise ratios.

1. Optimally weighted PCM system performance improves with code length, whereas unweighted PCM system performance is constant for $n \geqslant 5$.

2. For a code word length of $n = 10$, and moderate input rf signal-to-noise ratios, the rms analog error is reduced by about an order of magnitude.

3. For large signal-to-noise ratios the quantization error is dominant and therefore weighted and unweighted PCM systems are equal in performance.

4. The analog error due to channel noise is increased only slightly at low signal-to-noise ratios when double errors per code word are included.

Fig. 4 gives the relative pulse amplitudes required for each pulse position in an optimally weighted PCM word as a function of signal-to-noise ratio. Since received signal-to-noise ratio can be expected to vary with time, it is not feasible to achieve this optimal weighting continuously unless very elaborate equipment is designed. An alternative is the design of a near-optimum system in which the pulse weightings are kept fixed at the ideal values associated with some mean signal-to-noise ratio. In Fig. 5, modified values for the a_i's are given for the case where the pulse weightings are held fixed at the ideal values associated with the signal-to-noise ratio for which the analog error due to channel noise is equal to the quantization error. Thus for a code-word length $n = 10$, this occurs at $P/2\sigma^2 = 9.4$ db. The performance of this near-optimum system is shown in Figs. 6 and 7. It is seen that, over a wide range of moderate signal-to-noise ratios, most of the advantage due to weighting is maintained. At signal-to-noise ratios removed from the design value of 9.4 db, the weighted performance approaches that of the unweighted PCM system.

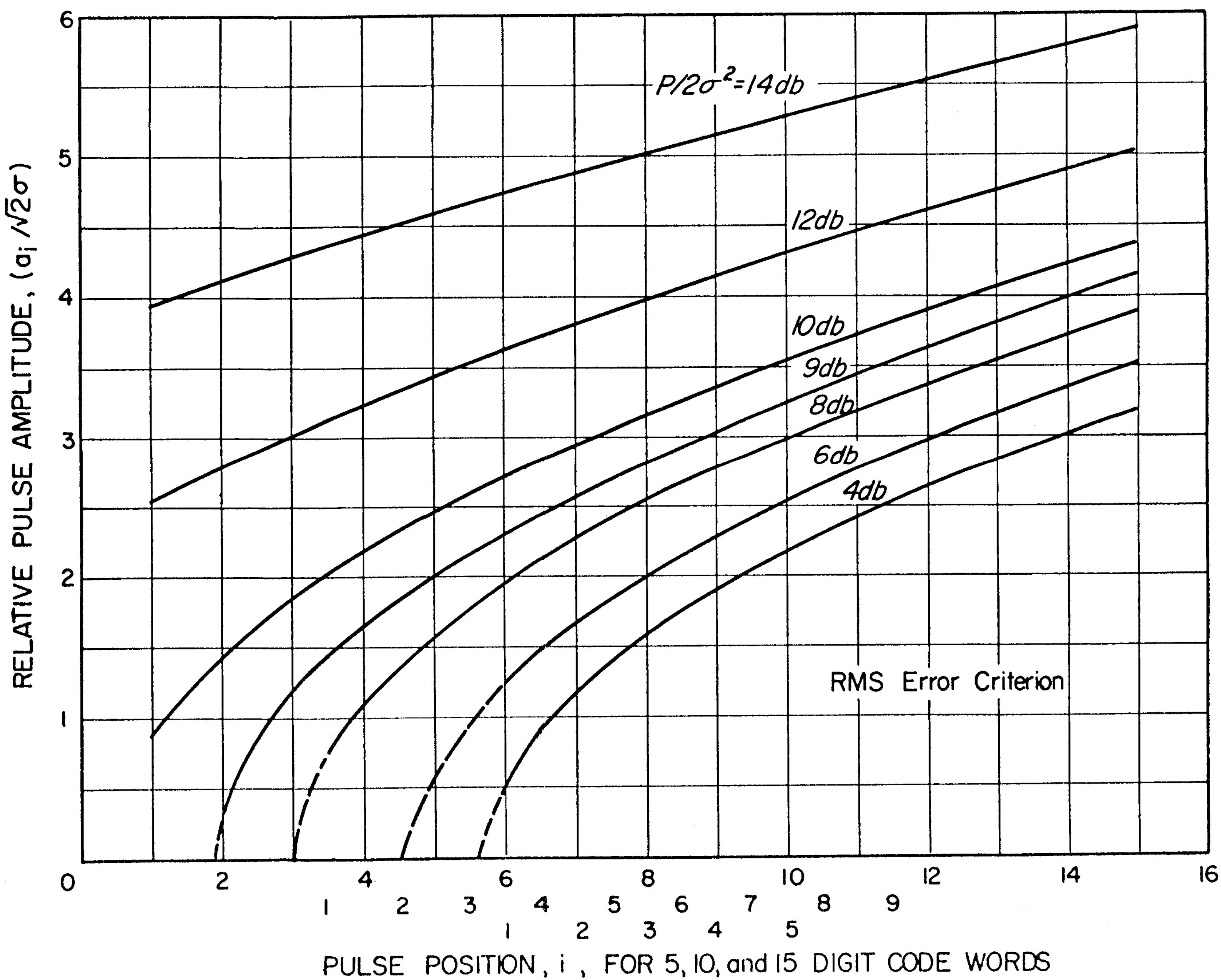

Fig. 4 Relative pulse amplitudes for an optimally weighted PCM system for several code lengths and signal-to-noise ratios.

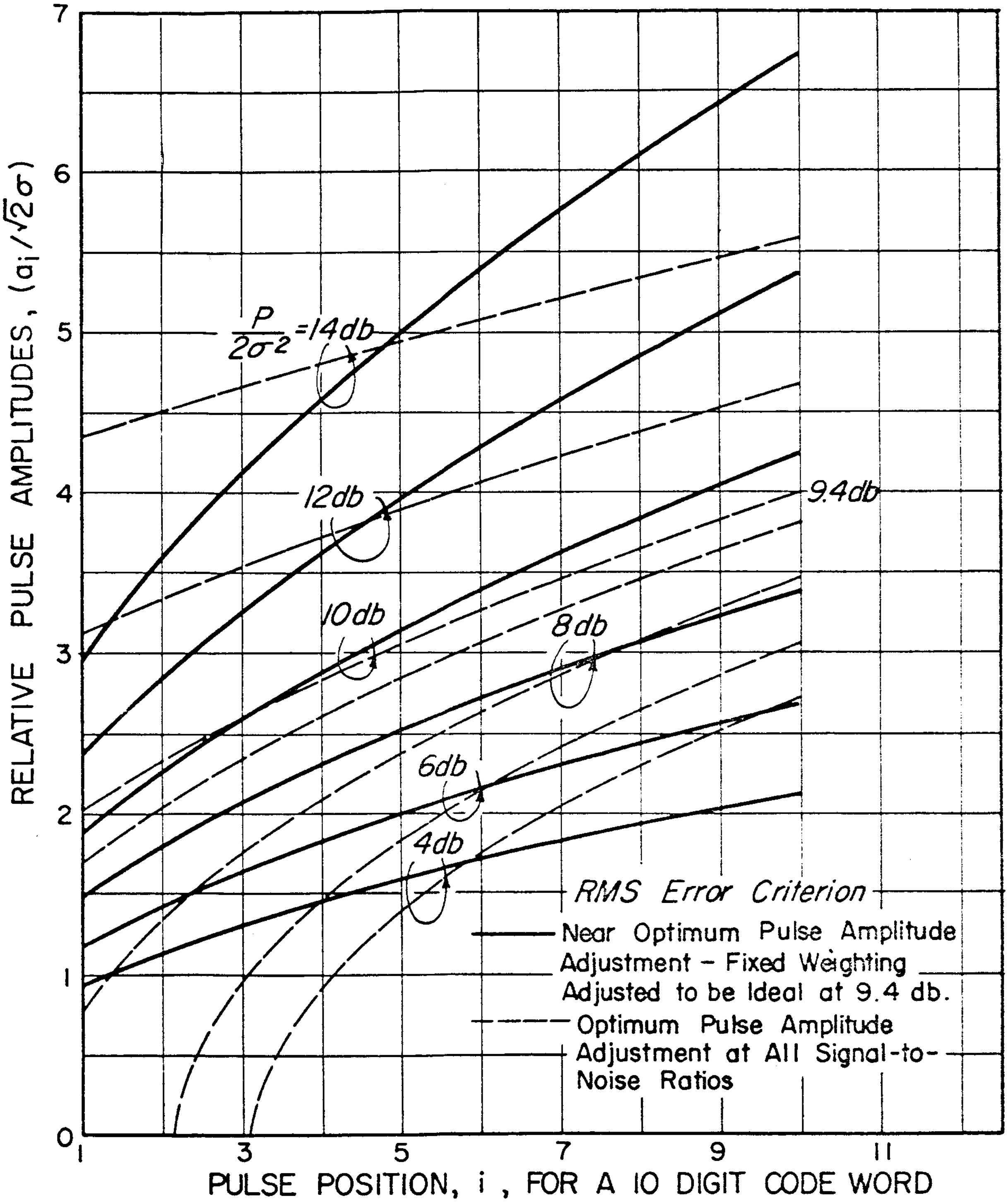

Fig. 5 Comparison of relative pulse amplitudes for near-optimum and optimum weighted systems for a code length n=10, and several signal-to-noise ratios.

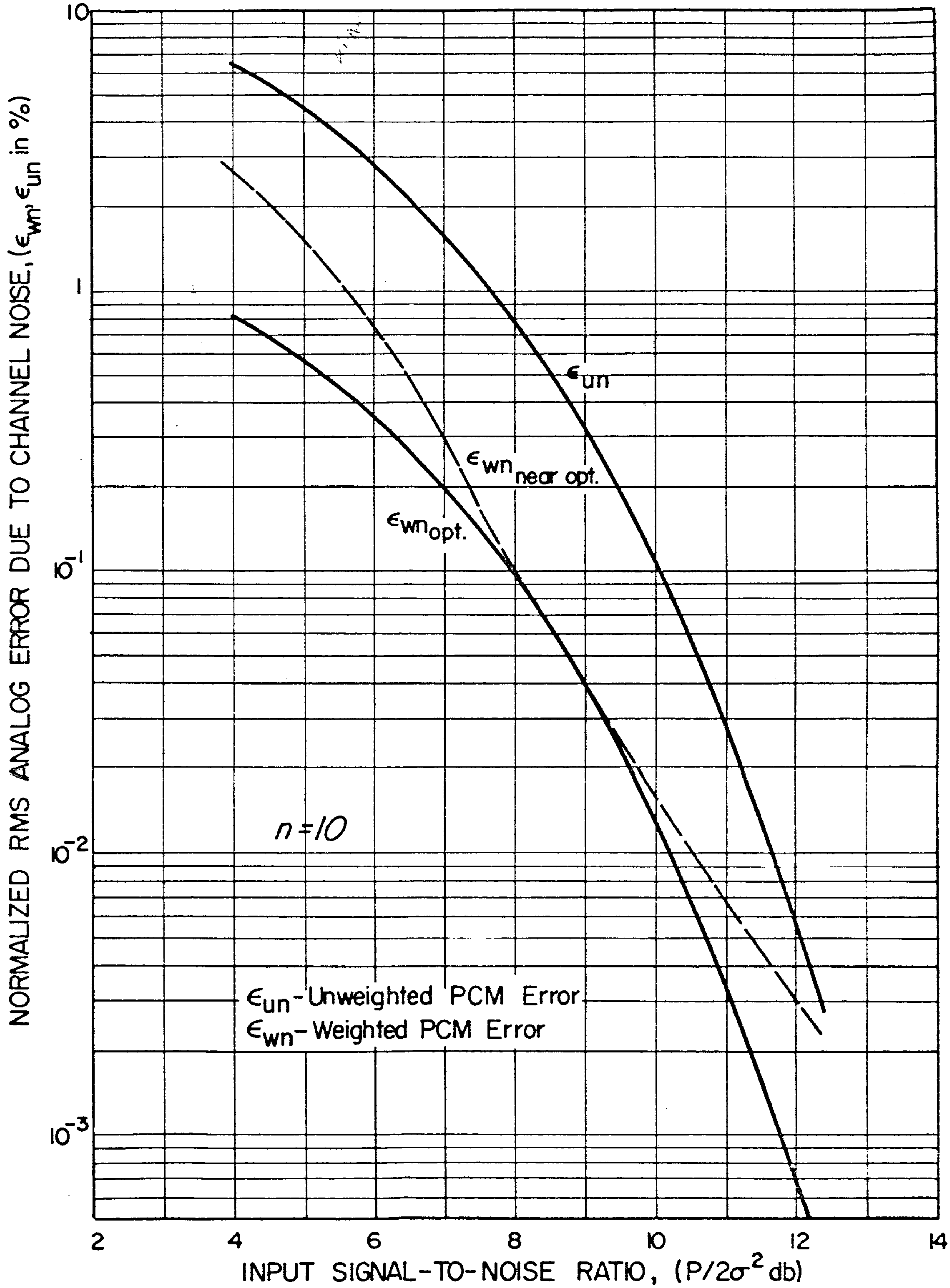

Fig. 6 Comparison of normalized rms analog errors, due to channel noise, of optimum and near-optimum weighted systems for a code length n=10 and several signal-to-noise ratios.

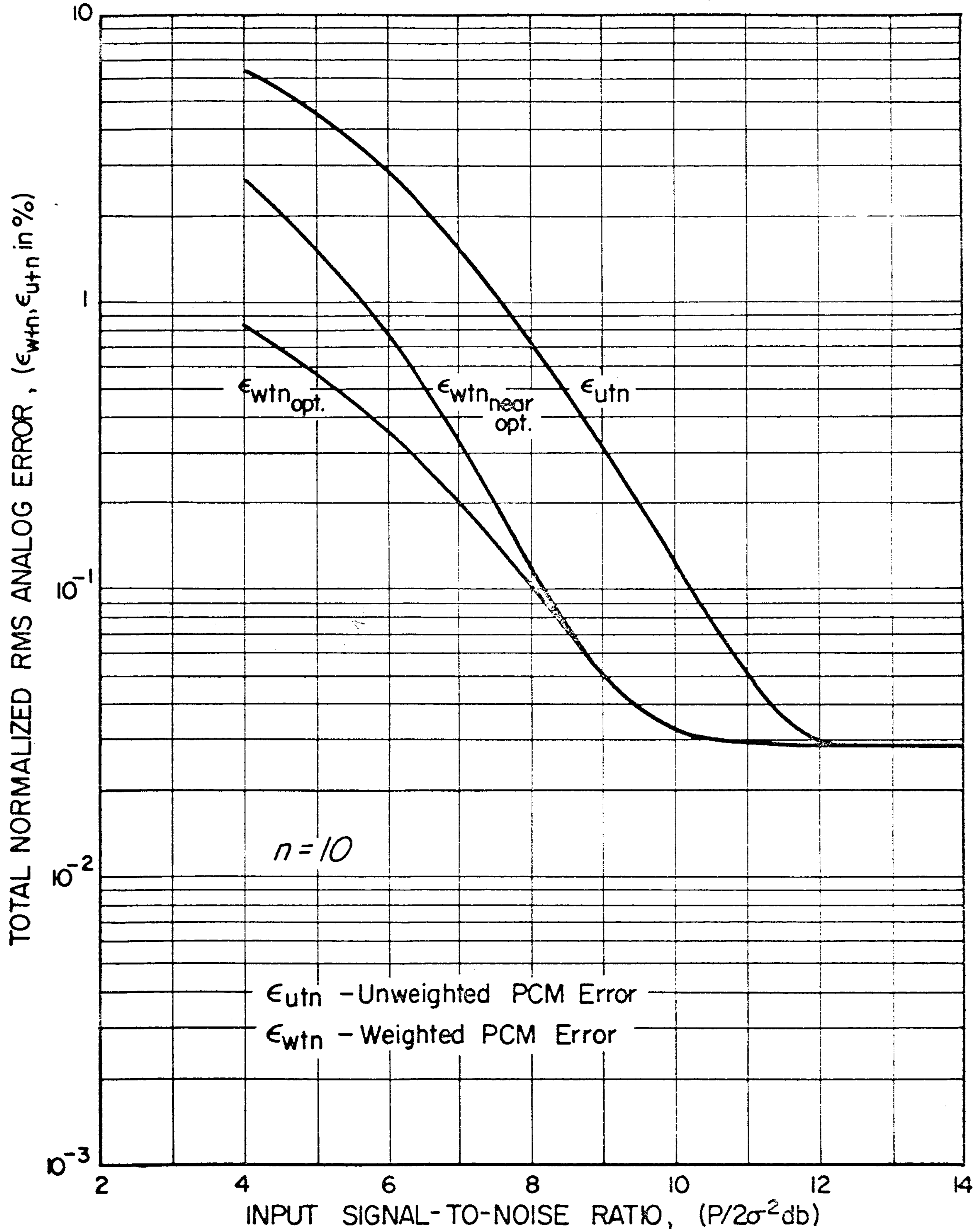

Fig. 7 Comparison of total normalized rms analog errors of optimum and near-optimum weighted systems for a code length n=10 and several signal-to-noise ratios.

A final set of curves in Fig. 8 give the results obtained by choosing a different performance criterion, namely - mean absolute analog error due to channel noise. Comparison with the rms analog error criterion is also shown. Fig. 9 indicates the required pulse amplitudes for this case.

Conclusion

Weighted PCM is a scheme which utilizes channel resources efficiently for the preservation of the fidelity or accuracy of data. In contrast to the normal viewpoint of minimizing digit error probability, a weighted PCM system is designed to favor the communication reliability of important portions of messages. It is shown to be capable of improving performance without the use of redundancy or error detecting and correcting coding.

The attainable improvement over an ordinary PCM system is a function of code length and signal-to-noise ratio. For example, at an average rf input signal-to-noise ratio of 8 db, and code length of 10 binary digits, the total normalized rms analog error is reduced from about 0.80 percent to 0.01 percent by optimally weighting the pulse amplitudes in the PCM word. This is equivalent to an effective signal-to-noise ratio gain of 2 db. Greater improvement is attained with the use of longer code lengths. Moreover, the analysis of a more practical weighted PCM system, wherein the pulse weightings are held fixed at the ideal weightings associated with a mean signal-to-noise ratio, shows that system performance is degraded only slightly over a wide range of channel S/N ratios.

The required pulse weightings may be obtained by varying power to different orders, or by effectively increasing the received signal-to-noise ratio by adjusting the bandwidths and/or time of the individual pulses.

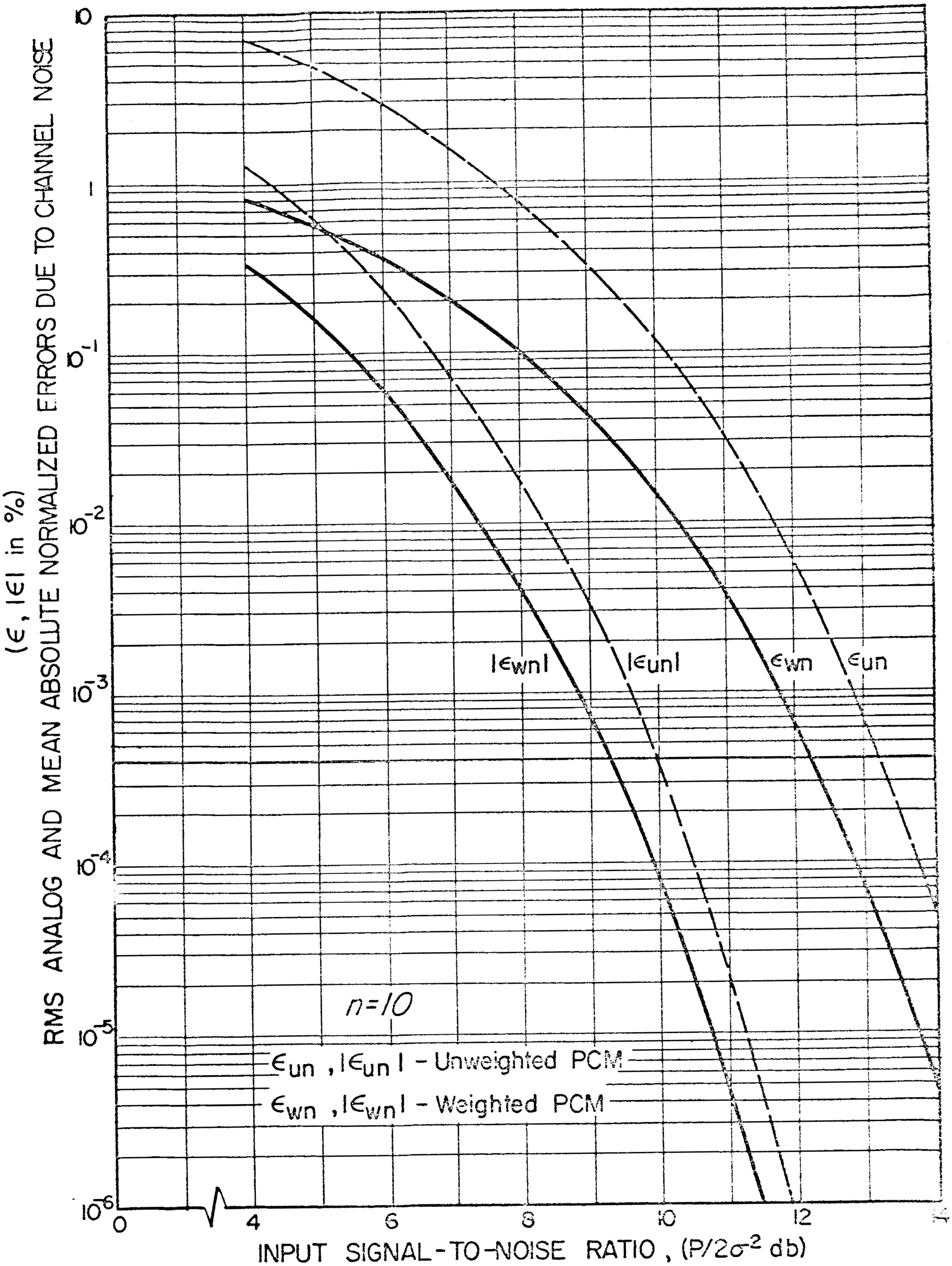

Fig. 8 Comparison of normalized mean absolute analog errors, due to channel noise, of weighted and unweighted PCM systems for a code length n=10, and several signal-to-noise ratios.

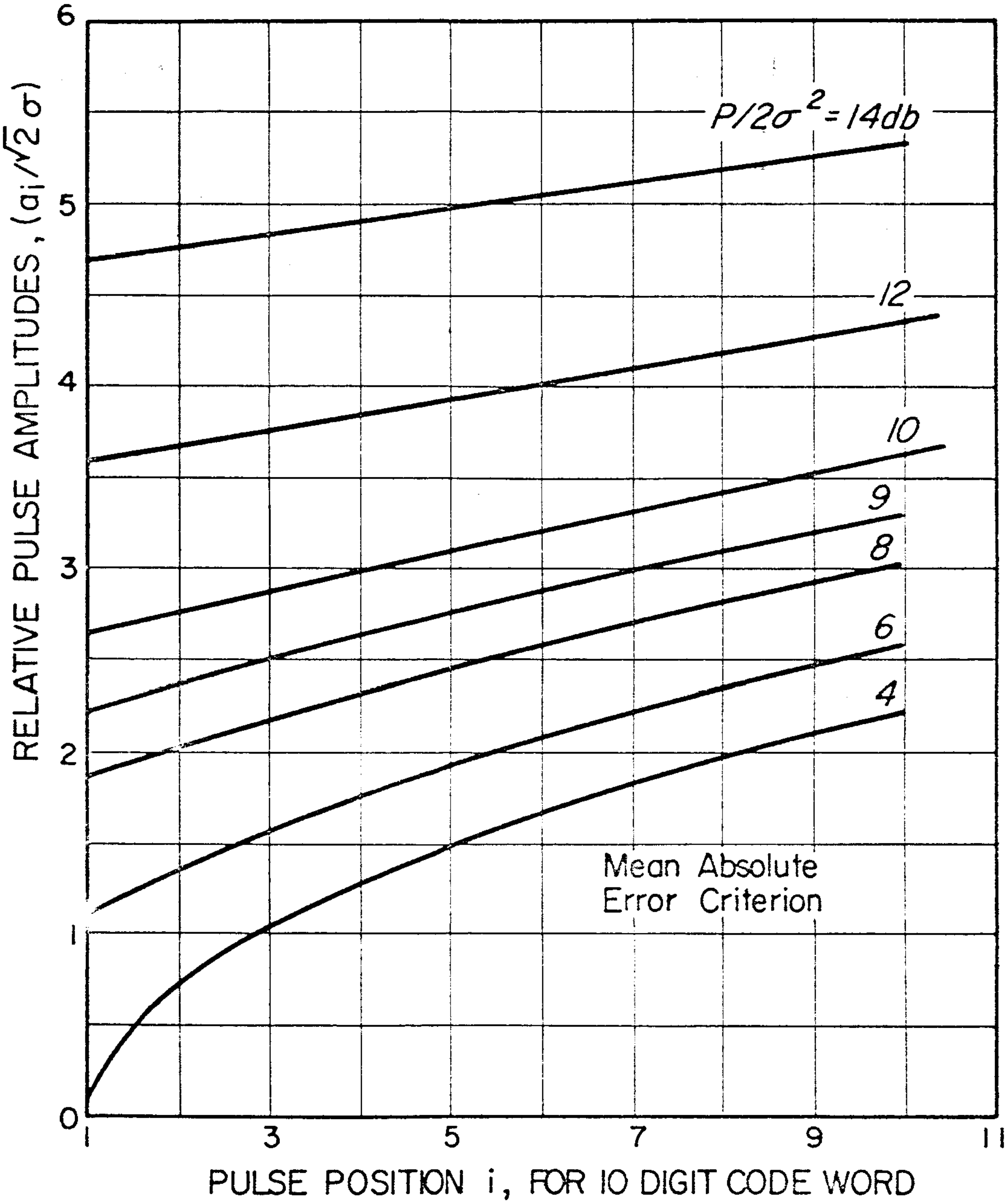

Fig. 9. Relative pulse amplitudes for an optumally (mean absolute error criterion) weighted PCM system for a code length n=10 and several signal-to-noise ratios.

In all schemes, the pulse parameters - power, bandwidth, and time, should be limited by some constraint on the <u>total</u> amounts of the available resources. Moreover, a similar weighting effect may be achieved by employing variable null-zone reception and decision feedback. The thresholds establishing the null-zone width may be set increasingly further apart as the order of the pulse in the code sequence increases. In this manner, increased reliability is obtained at the expense of increased pulse transmission time.

Bibliography

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2. E. Bedrosian, "Weighted PCM", PGIT - 4 No. 1, March 1958 p. 45.

3. New York University, "Evaluation and Optimization of Digital Communication Systems" Part II Second Scientific Report, Contract AF 19(604)-1964, July 15, 1957, AFCRC-TN-57-555(II), AD 117272.

GLOSSARY

a_i	Amplitude of ith order pulse, $i = 1,2, --- n$.
$A_m(n)$	Maximum analog range of quantized samples per unit quantization interval $= Q_m/q = 2^n - 1$.
b_o, b_1	Bias voltages.
d_j	Decimal equivalent of jth coded word $= \sum_{i=1}^{n} x_i(j) 2^{i-1}$; $x_i(j) = 0,1$.
e_i	Analog error of ith order pulse, $\pm q2^{i-1}$.
$\lvert e_i \rvert$	Absolute analog error of ith order, $\lvert q2^{i-1} \rvert$.
e_j	Analog error of jth order pulse, $\pm q2^{j-1}$.
$\widehat{e_d}$	Maximum analog error of both ith and jth pulse, $\pm q(2^{j-1} + 2^{i-1})$.
ϵ_q	RMS analog error due to quantization, $q/2\sqrt{3}$.
ϵ_u	RMS analog error of a pulse group due to channel noise in an unweighted PCM system.
$\lvert \epsilon_u \rvert$	Mean absolute analog error of a pulse group due to channel noise in an unweighted PCM system.
ϵ_w	RMS analog error of a pulse group due to channel noise in a weighted PCM system.
$\lvert \epsilon_w \rvert$	Mean absolute analog error of a pulse group due to channel noise in a weighted PCM system.
ϵ_{un}	RMS analog error of a pulse group due to channel noise normalized to Q_m in an unweighted PCM system.
$\lvert \epsilon_{un} \rvert$	Mean absolute analog error of a pulse group due to channel noise normalized to Q_m in an unweighted PCM system.

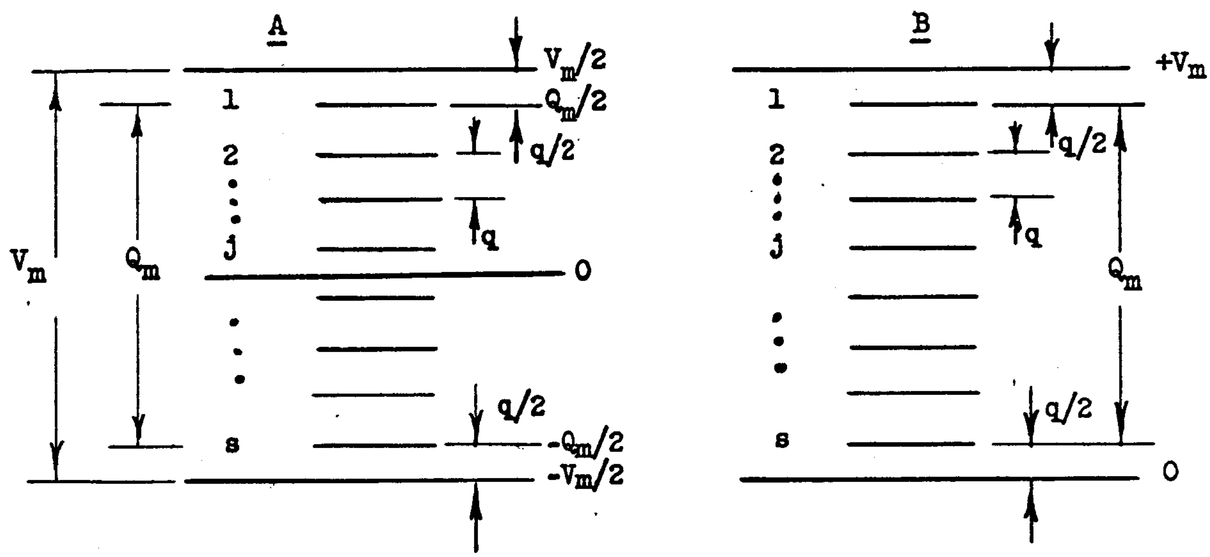
ϵ_{ut}	Total rms analog error of a pulse group in an unweighted PCM system.
ϵ_{utn}	Total rms analog error of a pulse group normalized to Q_m in an unweighted PCM system.
ϵ_{wn}	RMS analog error of a pulse group due to channel noise normalized to Q_m in a weighted PCM system.
$\lvert\epsilon_{wn}\rvert$	Mean absolute analog error of a pulse group due to channel noise normalized to Q_m in a weighted PCM system.
ϵ_{wt}	Total rms analog error of a pulse group in a weighted PCM system.
ϵ_{wtn}	Total rms analog error of a pulse group normalized to Q_m in a weighted PCM system.
ϵ'_u	RMS analog error of a pulse group due to channel noise in an unweighted PCM system including single and double errors per pulse group.
ϵ'_w	RMS analog error of a pulse group due to channel noise in a weighted PCM system including single and double errors per pulse group.
n	Word length of pulse group (quantized sample) in binary digits.
p	Digit error probability in an unweighted PCM system.
p_i	Error probability of ith digit in a weighted PCM system.
$p_i(1,n)$	Probability of single error in the ith position of a n-digit word for a weighted PCM system.
$p_{ij}(2,n)$	Probability of a double error in the ith and jth position of a n-digit word.
P	Average base-band signal power.
q	Amplitude of quantization interval.
Q_m	Peak-to-peak quantized signal range.
s	Number of quantized levels, $=2^n$.

v_j	Voltage of jth quantum level, $j = 1,2,----,s$.
V_m	Peak-to-peak signal amplitude.
w_i	Weight of ith order pulse, 2^{i-1}.
σ^2	Average channel noise power.
$\phi(a)$	Error integral, $\frac{2}{\sqrt{\pi}} \int_0^a e^{-x^2} dx$.

Appendix 1

Relations Between the Parameters of an Analog Signal and its Sampled-Quantized PCM Equivalent

The sketches shown below indicate the notation to be used for the two cases considered: A) the analog signal may assume both positive and negative values between $\pm V_m/2$ and B) the analog signal may assume only positive values between 0 and V_m.*



SYMMETRICAL CASE   UNSYMMETRICAL CASE

*See Glossary for definition of symbols, p. 21.

For the symmetrical case depicted in A, the jth quantum level v_j is

$$v_j(\pm) = \frac{Q_m}{2} - q(j-1) \qquad j = 1,2,---,s \tag{1}$$

Since $Q_m = (s-1)q$ and $s = 2^n$,

$$v_j(\pm) = \frac{Q_m}{2}\left[1 - \frac{2(j-1)}{s-1}\right] \tag{2}$$

or

$$v_j(\pm) = \frac{Q_m}{2}\left[1 - \frac{2(j-1)}{2^n-1}\right] \tag{3}$$

Moreover,

$$\begin{aligned} V_m &= Q_m + q \\ &= Q_m \frac{s}{s-1} = Q_m \frac{2^n}{2^n-1} \end{aligned} \tag{4}$$

Thus, (3) can be expressed in terms of V_m as

$$v_j(\pm) = \frac{V_m}{2^{n+1}}[2^n + 1 - 2j] \tag{5}$$

$$j = 1,2,---,s=2^n$$

For the assymmetrical case depicted in B

$$v_j(+) = v_j(\pm) + V_m/2 \tag{6}$$

Substituting (5) into (6) gives

$$v_j(+) = \frac{V_m}{2^{n+2}}[2^{n+2} + 2 - 4_j] \tag{7}$$

The maximum amplitude excursion of an n-digit PCM word is determined by taking the difference $v_1 - v_s$. Thus in (5) or (7)

$$v_1 - v_s = q(2^n - 1) \tag{8}$$

Defining $A_m(n)$ as the maximum analog range of the quantized samples per unit quantization interval,

$$A_m(n) = \frac{v_1 - v_s}{q}$$

$$= 2^n - 1 \tag{9}$$

Since $Q_m \overset{\Delta}{=} v_1 - v_s$

$$A_m(n) = Q_m/q \tag{10}$$

The relation between v_j and its coded equivalent is, for the symmetrical case,

$$v_j(\pm) = q\, d_j + b_o \quad , \tag{11}$$

$$v_j(\pm) = q \sum_{i=1}^{n} x_i(j)\, w_i + v_s(\pm) \tag{12}$$

where $$x_i(j) = 0,1 \tag{13}$$

and $$w_i = 2^{i-1}$$

It is presumed that the code increases from

$$00 \text{ --- } 0 \quad \text{for } v_s(\pm)$$

to

$$11 \text{ --- } 1 \quad \text{for } v_1(\pm) \ .$$

Since from inspection of plot A or from (5)

$$v_s(\pm) = -\frac{q}{2}[s-1] = -\frac{q}{2}[2^n-1] \tag{14}$$

$$v_j(\pm) = q\left\{\sum_{i=1}^{n} x_i(j)w_i - 2^{n-1} + 2^{-1}\right\} \tag{15}$$

or

$$v_j(\pm) = \frac{V_m}{2^n}\left\{\sum_{i=1}^{n} x_i(j)w_i - 2^{n-1} + 2^{-1}\right\} \tag{16}$$

For the asymmetrical case, the relation between $v_j(+)$ and its coded equivalent is

$$v_j(+) = q\ d_j + b_1 \tag{17}$$

$$= q\ d_j + v_s(+) \tag{18}$$

$$= q\ d_j + q/2 \tag{19}$$

$$= q\left\{\sum_{i=1}^{n} x_i(j)w_i + 2^{-1}\right\} \tag{20}$$

or

$$v_j(+) = \frac{V_m}{2^n}\left\{\sum_{i=1}^{n} x_i(j)w_i + 2^{-1}\right\} \ ; \ x_i(j) = 0,1 \tag{21}$$

It is again presumed that the code correspondence is

$$00 \text{ --- } 0 \text{ for } v_s(+)$$

$$11 \text{ --- } 1 \text{ for } v_1(+)$$

From (15) and (20) the error in amplitude resulting from noise on the telemetry channel is seen to be determined merely by the manner in which the received $x_i(j)$'s differ from the transmitted $x_i(j)$'s. Letting $v_j{*}$ be the received voltage level, the analog error per signal sample when transmitted by PCM is from (15) or (20)

$$v_j - v_j{*} = q(d_j - d_j{*}) \tag{22}$$

If d_j differs from $d_j{*}$ due to an error in the ith digit of their corresponding coded words, then

$$e_i = v_j - v_j{*}(i) = \pm\, q\, w_i = \pm\, q\, 2^{i-1} \tag{23}$$

Similarly if two digit errors per PCM word occur, the <u>maximum</u> analog error is

$$e_d = e_i + e_j = \pm\, q[2^{j-1} + 2^{i-1}] \tag{24}$$

As is shown in Appendix 3, the inclusion of (24) into the determination of rms analog error, increases the error only slightly. This justifies using (24) for finding an upper bound on ϵ_w' and eliminates the need for considering the various combinations of two-digit errors and their equivalent analog errors.

Appendix 2

Optimum Weighting of Individual Pulse Amplitudes in an n-Digit PCM Word for Minimum RMS Analog Error Subject to an Average Power Limitation.

The mean squared analog error due to channel noise of a pulse group or word in a weighted PCM system is

$$\epsilon_w^2 = \sum_{i=1}^{n} p_i(1,n) e_i^2 \tag{1}$$

if only single digit errors per word are considered. Above the system threshold $p_i(1,n)$ may be approximated by

$$p_i(1,n) = p_i \prod_{\substack{j=1 \\ j \neq i}}^{n} (1-p_j) \approx p_i \tag{2}$$

Since

$$e_i = \pm q\, 2^{i-1} \tag{3}$$

or

$$e_i^2 = q^2\, 4^{i-1} \tag{4}$$

(1) becomes

$$\epsilon_w^2 = q^2 \sum_{i=1}^{n} p_i\, 4^{i-1} \tag{5}$$

The average video signal power of a weighted PCM word is

$$P = \frac{1}{n} \sum_{i=1}^{n} a_i^2 \tag{6}$$

Hence, the problem reduces to finding the a_i's which will minimize (5) subject to the constraint (6). This problem may be solved using Lagrange's method of undetermined multipliers. If λ is the undetermined multiplier, it is necessary to find the appropriate set of a_i's from the (n+1) equations

$$\frac{\partial F}{\partial a_i} = 0 \qquad i = 1,2,--,n \tag{7}$$

and

$$g = 0 \tag{8}$$

where

$$F = F(a_1,a_2,---,a_n, \lambda) = f(a_1,a_2,---,a_n) + \lambda g(a_1,a_2---a_n) \tag{9}$$

$$f(a_1,a_2,---,a_n) = \epsilon_W^{\ 2} = q^2 \sum_{i=1}^{n} p_i 4^{i-1} \tag{10}$$

and

$$g(a_1,a_2,---,a_n) = P - \frac{1}{n}\sum_{i=1}^{n} a_i^{\ 2} \tag{11}$$

Moreover, for the system model assumed, i.e. a symmetric binary channel with additive gaussian noise and coherent detection

$$p_i = \frac{1}{\sqrt{2\pi}\,\sigma} \int_{a_i}^{\infty} e^{-n^2/2\sigma^2}\, dn \tag{12}$$

Thus from (7) through (12)

$$q^2\, 4^{i-1} \frac{dp_i}{da_i} - \frac{2}{n} a_i\lambda = 0 \ ; \quad i = 1,2,---,n \tag{13}$$

where
$$\frac{dp_i}{da_i} = - \frac{1}{\sqrt{2\pi}\,\sigma} \exp - \left(\frac{a_i^2}{2\sigma^2}\right) \tag{14}$$

Combining (13) and (14)

$$q^2 4^{i-1} \exp - \left(\frac{a_i^2}{2\sigma^2}\right) + \frac{2\sqrt{2\pi}\,\sigma\lambda}{n} a_i = 0; \; i=1,2,---,n \tag{15}$$

This equation may be solved for the a_i's most easily by the method of successive approximations. Noting that the first term in (15) is much more sensitive than the second term to variations in a_i, a first approximation is to set the first term equal to a constant, C. Since

$$q^2 4^{i-1} \exp - \left(\frac{a_i^2}{2\sigma^2}\right) = \exp - \left\{ \frac{a_i^2}{2\sigma^2} - (i-1)\ln 4 - \ln q^2 \right\} \tag{16}$$

then,

$$- \frac{a_i^2}{2\sigma^2} + (i-1)\ln 4 + \ln q^2 + C = 0$$

or
$$a_i^2 = (i-1)\,\sigma^2 \ln 16 + K \tag{17}$$

with
$$K = 2\,(C + \ln q^2)\sigma^2$$

From (11) and the identity

$$\sum_{i=1}^{n} (i-1) = \frac{n(n-1)}{2} \tag{18}$$

$$\sum_{i=1}^{n} a_i^2 = nP = \frac{n(n-1)}{2} \sigma^2 \ln 16 + nK$$

or

$$K = P - \frac{(n-1)}{2} \sigma^2 \ln 16 \tag{19}$$

Thus, substituting (19) into (17)

$$\frac{a_i^2}{\sigma^2} \doteq \frac{P}{\sigma^2} + [i - (\frac{n+1}{2})] \ln 16 \qquad (20)$$

A closer approximation is obtained by substituting (20) into the second term of (15) only, and solving for the new a_i's. Hence

$$\frac{a_i^2}{2\sigma^2} = - \ln (k_1 \frac{a_i}{4^{i-1}}) \qquad (21)$$

where

$$k_1 = \frac{-2\sqrt{2\pi}\ \lambda \sigma}{n\ q^2}, \text{ a constant,} \qquad (22)$$

and

$$\frac{a_i^2}{\sigma^2} = k_2 - \ln [P + (i - \frac{n+1}{2})\ \sigma^2 \ln 16]$$

$$+(i - 1) \ln 16 \qquad (23)$$

with k_2 a constant. Utilizing (11) again, k_2 may be evaluated as

$$k_2 = P/\sigma^2 - \frac{(n-1)}{2} \ln 16 + \frac{1}{n} \sum_{i=1}^{n} \ln[P+(i-\frac{n+1}{2})\sigma^2 \ln 16] \qquad (24)$$

The a_i's, therefore, are from (23) and (24)

$$\frac{a_i^2}{\sigma^2} = \left\{ \frac{P}{\sigma^2} + (i - \frac{n+1}{2}) \ln 16 \right\}$$

$$- \ln[P+[i-(\frac{n+1}{2})]\sigma^2 \ln 16]$$

$$+ \frac{1}{n} \sum_{i=1}^{n} \ln[P+[i-(\frac{n+1}{2})]\sigma^2 \ln 16 \qquad (25)$$

After some logarithmic approximations and manipulation (25) reduces to

$$\frac{a_i^2}{\sigma^2} = \frac{P}{\sigma^2} + \frac{P/\sigma^2 - 1}{P/\sigma^2}\left[i - \left(\frac{n+1}{2}\right)\right] \ln 16 \qquad (26)$$

This expression is most accurate for parameters which meet the condition

$$\frac{\left(i - \frac{n+1}{2}\right)\ln 16}{P/\sigma^2} < 1 \qquad (27)$$

Equation (26) is slightly different than that reported by Bedrosian[2], where the correction factor for the second term is

$$\frac{P/\sigma^2}{P/\sigma^2 + 1}$$

rather than $$\frac{P/\sigma^2 - 1}{P/\sigma^2}$$

It is noted that these factors are approximately equal for most useable P/σ^2's.

Appendix 3

Increase in Analog Error due to Channel Noise with the Inclusion of the Effects of Double Errors in an n-Digit PCM word

In appendix 2 the minimum rms analog error was determined neglecting the effects of more than one error per PCM code word. Whereas the analysis for the determination of the optimum a_i's is difficult to perform if multiple errors per code word are included, it is relatively straightforward to ascertain how much their presence modifies the previous results. It is the purpose of this appendix to determine a modified relation for the rms analog error due to single pluse double errors, wherein the pulse amplitudes a_i are set equal to their optimum values computed in the single error analysis.

The probability of a double error in the ith and jth position of a n-digit word is

$$p_{ij}(2,n) = p_i p_j \prod_{\substack{k=1 \\ k \neq i,j}}^{n} (1-p_k) \qquad (1)$$

$$\approx p_i p_j \text{ above threshold} \qquad (2)$$

Therefore the mean squared analog error (maximum) of a pulse group due to channel noise becomes

$$\epsilon_w'^2 = \sum_i p_i(1,n)e_i^2 + \sum_i \sum_j p_{ij}(2,n)\hat{e}_d^2$$

$$= q^2\left\{\sum_{i=1}^{n} p_i 4^{i-1} + \sum_{\substack{i=1 \\ i\neq j}}^{n} \sum_{j=1}^{n} p_i p_j (2^{j-1}+2^{i-1})^2\right\} \qquad (3)$$

where e_d, the maximum analog error due to errors in both ith and jth pulse is*

$$\hat{e}_d = \pm q\,[2^{j-1} + 2^{i-1}] \qquad (4)$$

Rearranging (3) gives

$$\epsilon_w'^2 = q^2\left\{\sum_{i=1}^{n} p_i 4^{i-1} + \sum_{j=1}^{n-1} p_j \sum_{i=j+1}^{n} p_i [4^{i-1}+4^{j-1}+2^{i+j-1}]\right\} \quad (5)$$

The normalized rms error for weighted PCM is then

$$\epsilon_{wn}' = \frac{\epsilon_w'}{q(2^n-1)} \qquad (6)$$

For the unweighted or normal PCM case

$$\epsilon_u'^2 = \sum_i p_i(1,n)e_i^2 + \sum_i \sum_j p_{ij}(2,n)\hat{e}_d^2 \qquad (7)$$

where $\qquad p_{ij}\,(2,n) \approx p_i^2 = p^2$

and $\qquad p_i\,(1,n) \approx p$

*It is to be noted that the maximum effect of double errors has been assumed. The result (3) therefore gives an upper bound, which proves to be a tight bound.

Thus,

$$\epsilon_w'^2 = q^2 \left\{ p \sum_{i=1}^{n} 4^{i-1} + p^2 \sum_{j=1}^{n-1} \sum_{i=j+1}^{n} [4^{i-1}+4^{j-1}+2^{i+j-1}] \right\} \quad (8)$$

Summing the terms in (8) gives

$$\epsilon_u'^2 = q^2 \left\{ p \frac{4^n-1}{3} + p^2[(n + \frac{2}{3}) \frac{4^n}{3} + \frac{n(4^{n-1}-1)}{3} - 2^{n+1} + \frac{16}{9} + \sum_{j=1}^{n-1} j4^{j-1}] \right\} \quad (9)$$

The normalized rms error for unweighted PCM is then

$$\epsilon_{un}' = \frac{\epsilon_u'}{q(2^n-1)} \quad (10)$$

Appendix 4

Optimum Weighting of Individual Pulse Amplitudes in an n-Digit PCM Word for Minimum Mean Absolute Analog Error Subject to an Average Power Limitation

The mean absolute analog error due to channel noise of a pulse group or word in a weighted PCM system is

$$|\epsilon_w| = \sum_{i=1}^{n} p_i(1,n)\,|e_i| \tag{1}$$

if only single digit errors per word are considered. Since

$$p_i(1,n) \approx p_i$$

and

$$|e_i| = q2^{i-1}$$

$$|\epsilon_w| = q\sum_{i=1}^{n} p_i 2^{i-1} \tag{2}$$

In an analogous manner to the derivation in Appendix 2 , with the only change being

$$f(a_1, a_2, \cdots, a_n) = |\epsilon_w| = q\sum_{i=1}^{n} p_i 2^{i-1} \tag{3}$$

it can be readily shown that the optimum a_i's are defined by

$$\frac{a_i^2}{\sigma^2} = \frac{p}{\sigma^2} + \frac{p/\sigma^2 - 1}{p/\sigma^2}\left[i - \frac{n+1}{2}\right]\ln 4 \tag{4}$$